

TIAN'S INVARIANT OF THE GRASSMANN MANIFOLD

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ABSTRACT. — We prove that Tian's invariant on the complex Grassmann manifold $G_{p,q}(\mathbb{C})$ is equal to $1/(p+q)$. The method introduced here uses a Lie group of holomorphic isometries which operates transitively on the considered manifolds and a natural imbedding of $(\mathbb{P}^1(\mathbb{C}))^p$ in $G_{p,q}(\mathbb{C})$.

RÉSUMÉ. — On prouve que l'invariant de Tian sur la grassmannienne $G_{p,q}(\mathbb{C})$ est $1/(p+q)$. La méthode présentée dans cet article utilise un groupe de Lie d'isométries holomorphes qui opère transitivement sur les variétés considérées ainsi qu'un plongement naturel de $(\mathbb{P}^1(\mathbb{C}))^p$ dans $G_{p,q}(\mathbb{C})$.

1. INTRODUCTION

On a complex manifold, an hermitian metric h is characterized by the *1-1 symplectic* form ω defined by $\omega = i g_{\lambda\bar{\mu}} dz^\lambda \wedge d\bar{z}^\mu$, where $g_{\lambda\bar{\mu}} = h_{\lambda\bar{\mu}}/2$.

The metric is a *Kähler metric* if ω is closed, i. e. $d\omega = 0$; then M is a *Kähler manifold*. On a Kähler manifold, we can define the *Ricci form* by $R = i R_{\lambda\bar{\mu}} dz^\lambda \wedge d\bar{z}^\mu$, where $R_{\lambda\bar{\mu}} = -\partial_{\lambda\bar{\mu}} \log |g|$.

A Kähler manifold is *Einstein with factor k* if $R = k\omega$. For instance, choosing a local coordinate system $Z = (z_1, \dots, z_m)$, the projective space $\mathbb{P}_m(\mathbb{C})$ with the Fubini-Study metric $\omega = i\partial\bar{\partial} \log(1 + \|Z\|^2)$ is Einstein with factor $m+1$.

On a Kähler manifold M , the *first Chern class* $C^1(M)$ is the cohomology class of the Ricci tensor, that is the set of the forms $R + i\partial\bar{\partial}\varphi$, where φ is C^∞ on M . If there is a form in $C^1(M)$ which is positive (resp. negative, zero), then $C^1(M)$ is *positive* (resp. *negative*, *zero*). If a Kähler manifold is Einstein, then $C^1(M)$ and k are both positive (resp. negative, zero). In the negative case, it was proved by Aubin ([Au1], see also [Au4]), that there exists a unique Einstein-Kähler metric (E.K. metric) on M . It is so for the zero case too ([Au1], [Ya]). The question for the positive case is still open: some manifolds, such as the complex projective space blown up at one point, do not admit an E.K. metric (for obstructions, see [Li] and [Fu]). Aubin [Au2] and Tian [Ti] have shown that for suitable values of holomorphic invariants of the metric, there exists an E.K. metric on M .

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For $\omega/2\pi$ in $C^1(M)$, *Tian's invariant* $\alpha(M)$ is the supremum of the set of the real numbers α satisfying the following: there exists a constant C such that the inequality $\int_M e^{-\alpha\varphi} \leq C$ holds for all the C^∞ functions φ with $\omega + i\partial\bar{\partial}\varphi > 0$ and $\sup \varphi \geq 0$, where $\omega = i g_{\lambda\bar{\mu}} dz^\lambda \wedge d\bar{z}^\mu$ is the metric form. Such functions φ are said *ω -admissible*.

In [Ti], Tian established that if $\alpha(M) > m/(m+1)$, m being the dimension of M , there exists an E.K. metric on M . This condition is not necessary: it does not hold on the projective space, where Tian's invariant is $1/(m+1)$.

In the same paper, Tian introduces a more restrictive invariant $\alpha_G(M)$, considering only the admissible functions φ invariant by the action of a compact group G of holomorphic isometries. The sufficient condition for the existence of an E.K. metric on M remains $\alpha_G(M) > m/(m+1)$; it is more easily satisfied if the group G is rich enough.

In many cases, the group G is a non-discrete Lie group. The invariant $\alpha_G(M)$ can be computed using subharmonic functions methods and the maximum principle (for effective examples, see [Be1], [Be2], [Be-Ch1], [Be-Ch2], [Re]).

In this paper, we prove the following theorem:

Theorem 1.1. *Tian's invariant on $G_{p,q}(\mathbb{C})$ is given by $\alpha(G_{p,q}(\mathbb{C})) = 1/(p+q)$.*

This generalizes the known result on $\mathbb{P}^m(\mathbb{C})$ ([Ti], see also [Au3]). Let us also mention that Tian's invariant has been computed on $\mathbb{P}^m(\mathbb{C})$ blown up at one point and on certain Fermat hypersurfaces using Hörmander L^2 estimates for the $\bar{\partial}$ -equation ([Ti]).

We first compute the volume element of the metric $\mathcal{G}_{p,q}$; then we will establish some general preliminary results concerning Tian's invariant as well as imbeddings of $\{\mathbb{P}^1(\mathbb{C})\}^p$ in $G_{p,q}(\mathbb{C})$ which allow us to deduce $\alpha(G_{p,q}(\mathbb{C}))$ from $\alpha(\mathbb{P}^1(\mathbb{C}))$.

2. BASIC PROPERTIES OF THE GRASSMANN MANIFOLD

We propose here a short survey of the properties of Grassmann manifold (for more details, see [Ko-No]). We denote by $G_{p,q}(\mathbb{C})$ the set of the subspaces of dimension p in \mathbb{C}^{p+q} ; in particular, $G_{1,m}(\mathbb{C})$ is the complex projective space of dimension m . It is known (see [Au3]) that on $\mathbb{P}_m(\mathbb{C})$, the Fubini-Study metric is Einstein with factor $m+1$ and that Tian's invariant is $1/(m+1)$. Now, let $M^*(p+q, p)$ be the set of the matrices of rank p in $M_{p+q,p}(\mathbb{C})$. The group $Gl_p(\mathbb{C})$ acts by multiplication on the right on $M^*(p+q, p)$. More precisely $(M^*(p+q, p), \pi, G_{p,q}(\mathbb{C}))$ is a principal fiber bundle with group $Gl_p(\mathbb{C})$. The group $Gl_{p+q}(\mathbb{C})$ acts by multiplication on the left on $M^*(p+q, p)$ and induces an action on $G_{p,q}(\mathbb{C})$; so does the unitary group $U(p+q)$. These groups act transitively on $G_{p,q}(\mathbb{C})$, which shows that $G_{p,q}(\mathbb{C})$ is compact.

We denote by \mathcal{I} the set of all increasing-ordered subsets of p elements in $\{1, \dots, p+q\}$. Let P be an element of $M^*(p+q, p)$, $P = (p_{ij})_{\substack{1 \leq i \leq p+q \\ 1 \leq j \leq p}}$. By Cauchy-Binet formula we get: $\det({}^t P \bar{P}) = \sum_{I \in \mathcal{I}} |\det m_I(P)|^2$, where $m_I(P)$ is the matrix $(p_{ij})_{\substack{i \in I \\ 1 \leq j \leq p}}$. The form ω , where $\omega = i \partial \bar{\partial} \log \det({}^t P \bar{P})$, is invariant by the action of $Gl_p(\mathbb{C})$ on $M^*(p+q, p)$, and so it projects onto a form $\mathcal{G}_{p,q}$. The metric $\mathcal{G}_{p,q}$ is a Kähler metric form on $G_{p,q}(\mathbb{C})$. For $p = 1$, this metric on $G_{1,m}(\mathbb{C})$ is the Fubini-Study metric on the complex projective space. The action of the unitary group $U(p+q)$ on $G_{p,q}(\mathbb{C})$ preserves the metric $\mathcal{G}_{p,q}$ so that $U(p+q)$ is a group of holomorphic isometries which operates transitively on $G_{p,q}(\mathbb{C})$. For I in \mathcal{I} , let U_I be the set of the matrices P in $M^*(p+q, p)$ such that $\det(m_I(P))$ is non-zero. Then $\pi(U_I)$ is a coordinate open set on $G_{p,q}(\mathbb{C})$, the matrix Z_I in $M_{q,p}(\mathbb{C})$ is the coordinate, the inverse of the chart φ_I sends $M^*(p+q, p)$ onto $\pi(U_I)$ and we have $m_I(\varphi_I^{-1}(Z_I)) = I^{(p)}$ where $I^{(p)}$ is the $p \times p$ identity matrix, and $m_{I^c}(\varphi_I^{-1}(Z_I)) = Z_I$.

Lemma 2.1. *For I in \mathcal{I} , let λ_I be the map from $\pi(U_I)$ to \mathbb{R}_+ defined by*

$$\lambda_I(Z_I) = \left| \det(\text{Id} + {}^t Z_I \bar{Z}_I) \right|^{-(p+q)}.$$

Then $(\lambda_I)_{I \in \mathcal{I}}$ are the components of a maximal differential form η on $G_{p,q}(\mathbb{C})$, namely:

$$\eta = \lambda_I (i/2)^{pq} (dZ \wedge d\bar{Z})_I.$$

Proof. It suffices to show that the following transformation rule holds:

$$\text{for every } I, \tilde{I} \text{ in } \mathcal{I}, \lambda_I \text{ is equal to } \lambda_{\tilde{I}} \times \left| \det \frac{\partial Z_{\tilde{I}}}{\partial Z_I} \right|^2 \text{ on } \pi(U_I) \cap \pi(U_{\tilde{I}}).$$

Let P_I be the matrix $\varphi_I^{-1}(Z_I)$. Then $P_I \{m_{\tilde{I}}(P_I)\}^{-1} = P_{\tilde{I}}$, so $Z_{\tilde{I}} = m_{\tilde{I}^c}(P_I) \{m_{\tilde{I}}(P_I)\}^{-1}$. The differential of the map which sends Z_I on P_I is the map which sends H on \check{H} , where $m_{I^c}(\check{H}) = H$ and $m_I(\check{H}) = 0$. The change of charts sending Z_I on $Z_{\tilde{I}}$, we obtain

$$\begin{aligned} D Z_{\tilde{I}}(H) &= m_{\tilde{I}^c}(\check{H}) \{m_{\tilde{I}}(P_I)\}^{-1} - m_{\tilde{I}^c}(P_I) \{m_{\tilde{I}}(P_I)\}^{-1} m_{\tilde{I}}(\check{H}) \{m_{\tilde{I}}(P_I)\}^{-1} \\ &= (m_{\tilde{I}^c}(\check{H}) - \gamma m_{\tilde{I}}(\check{H})) \alpha^{-1}, \end{aligned}$$

$$\text{where } \alpha = m_{\tilde{I}}(P_I), \beta = m_{\tilde{I}^c}(P_I) \text{ and } \gamma = \beta \alpha^{-1}.$$

Let us define a map u from $M_{q,p}(\mathbb{C})$ to $M_{q,p}(\mathbb{C})$ by $u(H) = m_{\tilde{I}^c}(\check{H}) - \gamma m_{\tilde{I}}(\check{H})$. We can choose $I = \{q+1, \dots, q+p\}$ and $\tilde{I} = \{1, \dots, r\} \cup \{q+1+r, \dots, q+p\}$, where $0 \leq r \leq \inf(p, q)$. We define the $k \times l$ matrix $E_{i,j}^{(k \times l)}$ by $(E_{i,j}^{(k \times l)})_{\lambda\mu} = \delta_{i\lambda} \delta_{j\mu}$. We have

$$m_{\tilde{I}}(\check{E}_{i,j}^{(q \times p)}) = E_{i,j}^{(p \times p)} \quad \text{if } i \leq r, \text{ and } 0 \text{ if } i > r,$$

$$\text{and } m_{\tilde{I}^c}(\check{E}_{i,j}^{(q \times p)}) = E_{i-r,j}^{(q \times p)} \quad \text{if } i > r, \text{ and } 0 \text{ if } i \leq r. \text{ Hence}$$

$$(\gamma m_{\tilde{I}}(\check{E}_{i,j}^{(q \times p)}))_{\alpha\beta} = \gamma_{\alpha i} m_{\tilde{I}}(\check{E}_{i,j}^{(q \times p)})_{ij} \delta_{j\beta} = \gamma_{\alpha i} \delta_{j\beta} \quad \text{if } i \leq r, \text{ and } 0 \text{ elsewhere.}$$

Now the map which sends H to $\gamma m_{\tilde{I}}(\check{H})$ can be restricted if $1 \leq j \leq p$ to the span B_j of the $(E_{i,j})_{1 \leq i \leq q}$. The r first columns of its matrix are those of γ , the others are 0. The map which sends H to $\gamma m_{\tilde{I}^c}(\check{H})$ maps also B_j into itself. The right upper block of its matrix is $I^{(q-r)}$, the other elements are 0. This allows us to compute the matrix of the restriction of u to B_j , whose determinant is $(-1)^{r \times (q-r)} \det(\gamma_{ij})_{\substack{q-r+1 \leq i \leq q \\ 1 \leq j \leq r}}$.

So $\det u = (-1)^{p \times r \times (q-r)} \left[\det(\gamma_{ij})_{\substack{q-r+1 \leq i \leq q \\ 1 \leq j \leq r}} \right]^p$. For $1 \leq i \leq q$, let C_i be the span of the $(E_{i,j})_{1 \leq j \leq p}$. Each C_i is stable by the map from $M_{q,p}(\mathbb{C})$ to $M_{q,p}(\mathbb{C})$ which sends H to $H \alpha^{-1}$. The matrix of the restriction is α^{-1} , so the determinant of the map is $(\det \alpha)^{-q}$. Hence

$$\left| \det DZ_{\tilde{I}}(H) \right|^2 = \left| \det(\gamma_{i,j})_{\substack{q-r+1 \leq i \leq q \\ 1 \leq j \leq r}} \right|^{2p} \times \left| \det \alpha \right|^{-2q}.$$

Let A be the right $r \times r$ upper block of α . The left $(p-r) \times (p-r)$ lower block of α is $I^{(p-r)}$ and the right $(p-r) \times r$ lower block is 0, so $\det \alpha = (-1)^{r(p-r)} \det A$. The left $r \times (p-r)$ lower block of β is 0, the right $r \times r$ block is $I^{(r)}$ so that the left $r \times r$ lower block of γ is A^{-1} .

From this we deduce $\left| \det DZ_{\tilde{I}}(H) \right|^2 = \left| \det \alpha \right|^{-2(p+q)}$. Since $P_I \alpha^{-1} = P_{\tilde{I}}$, we have

$$\lambda_{\tilde{I}} = \left| \det({}^t P_{\tilde{I}} \bar{P}_{\tilde{I}}) \right|^{-(p+q)} = \left| \det \alpha \right|^{2(p+q)} \lambda_I = \left| \det \frac{\partial Z_{\tilde{I}}}{\partial Z_I} \right|^{-2} \lambda_I.$$

□

Lemma 2.2. *The unitary group $U(p+q)$ preserves η .*

Proof. We call I the set $\{q+1, \dots, q+p\}$. We define P_I in $\pi(U_I)$ by $P_I = \varphi_I^{-1}(Z_I)$. Let U be an element in $U(p+q)$ such that $m_I(UP_I)$ is invertible. Let $\tilde{P}_I = UP_I \{m_I(UP_I)\}^{-1}$ and $\tilde{Z}_I = m_{I^c}(\tilde{P}_I)$. We have $\tilde{Z}_I = m_{I^c}(U) P_I \{m_I(U) P_I\}^{-1}$. So

$$D\tilde{Z}_I(H) = m_{I^c}(U) \left[\check{H} \{m_I(U) P_I\}^{-1} - P_I \{m_I(U) P_I\}^{-1} m_I(U) \check{H} \{m_I(U) P_I\}^{-1} \right].$$

Thus $D\tilde{Z}_I(H) = X \check{H} \delta^{-1}$, where $\delta = m_I(U) P_I$ and $X = m_{I^c}(U) [I^{(p+q)} - P_I \delta^{-1} m_I(U)]$. Let X_1 be the $q \times q$ matrix of the q first columns of X . Then, $X \check{H} = X_1 H$ and we get $D\tilde{Z}_I(H) = X_1 H \delta^{-1}$. The determinant of the map from $M_{q,p}(\mathbb{C})$ to $M_{q,p}(\mathbb{C})$ which sends H to $H \delta^{-1}$ is $(\det \delta)^{-q}$. The determinant of the map from $M_{q,p}(\mathbb{C})$ to $M_{q,p}(\mathbb{C})$ which sends H to $X_1 H$ is $(\det X_1)^p$, so $\det D\tilde{Z}_I = (\det X_1)^p (\det \delta)^{-q}$. We divide U into four blocks:

$$U = \begin{pmatrix} U_q & U_{q,p} \\ U_{p,q} & U_p \end{pmatrix}, \quad U_q \in M_q(\mathbb{C}), \quad U_p \in M_p(\mathbb{C}), \quad U_{p,q} \in M_{p,q}(\mathbb{C}), \quad U_{q,p} \in M_{q,p}(\mathbb{C}).$$

Then $\delta = U_{p,q} Z_I + U_p$, so $X_1 = U_q - (U_q Z_I + U_{q,p}) (U_{p,q} Z_I + U_p)^{-1} U_{q,p}$. Let Z in $M_{p+q,p+q}(\mathbb{C})$ be the matrix with blocks $Z_q = I^{(q)}$, $Z_{p,q} = 0$, $Z_{q,p} = Z_I$, $Z_p = I^{(p)}$, the notations being the same as above. Writing $\det U = \det(UZ)$ and using the column transformation $C_1 \leftarrow C_1 - C_2 (U_{p,q} Z_I + U_p)^{-1} U_{p,q}$ where C_1 is made of the first q columns and C_2 of the remaining ones, we get

$$\det U = \det \left[U_q - (U_q Z_I + U_{q,p}) (U_{p,q} Z_I + U_p)^{-1} U_{p,q} \right] \times \det(U_{p,q} Z_I + U_p).$$

Hence $|\det D\tilde{Z}_I|^2 = |\det \delta|^{-2(p+q)}$. We have $\tilde{P}_I = AP_I \delta^{-1}$, so

$$\lambda_{\tilde{I}} = \det({}^t \tilde{P}_I \bar{\tilde{P}}_I)^{-(p+q)} = \det({}^t P_I \bar{P}_I)^{-(p+q)} \times |\det \delta|^{2(p+q)} = \lambda_I |\det D\tilde{Z}_I|^{-2},$$

which proves the result. \square

Proposition 2.3. 1. $dV(\mathcal{G}_{p,q}) = \eta$.

2. If $I \in \mathcal{I}$, $\left| \mathcal{G}_{p,q} \right|_I = \left\{ \det(I^{(p)} + {}^t Z_I \bar{Z}_I) \right\}^{-(p+q)}$.

3. $\mathcal{R}(\mathcal{G}_{p,q}) = (p+q)\mathcal{G}_{p,q}$.

Proof. 1. Let I in \mathcal{I} . It is easy to compute $\mathcal{G}_{p,q}$ at the point $Z_I = 0$: $\mathcal{G}_{p,q}(H, K) = \text{Tr}(H\bar{K})$. Then $dV(\mathcal{G}_{p,q})|_{Z_I=0} = (i/2)^{pq} (dZ \wedge d\bar{Z})_I = \eta|_{Z_I=0}$. Since $dV(\mathcal{G}_{p,q})$ and η are invariant by the transitive action of $U(p+q)$, we have $dV(\mathcal{G}_{p,q}) = \eta$.

2. Since $dV(\mathcal{G}_{p,q}) = \left| \mathcal{G}_{p,q} \right|_I (i/2)^{pq} (dZ \wedge d\bar{Z})_I$, property 1 gives the result.

3. Remark that $\mathcal{G}_{p,q} = i \partial \bar{\partial} \log \{ \det(I^{(p)} + {}^t Z_I \bar{Z}_I) \}$. Since $\mathcal{R}(\mathcal{G}_{p,q}) = -i \partial \bar{\partial} \log \left| \mathcal{G}_{p,q} \right|_I$, we obtain $\mathcal{R}(\mathcal{G}_{p,q}) = (p+q)\mathcal{G}_{p,q}$, which expresses that $\mathcal{G}_{p,q}$ is Einstein, with factor $p+q$. \square

3. SOME GENERAL RESULTS ABOUT TIAN'S INVARIANT

3.1. Tian's invariant with a normalization on a finite set. If X is a manifold, we will denote by μ_X a measure on X compatible with the manifold structure.

Theorem 3.1. *Let M be a compact Kähler manifold. We suppose that there exists a compact Lie group G of holomorphic isometries. Let $\Delta_n = \{P_1, \dots, P_n\}$ be a finite subset of M . Let $\alpha(\omega)$ (resp. $\alpha_{\Delta_n}(\omega)$) be the supremum of the set of the nonnegative real numbers α satisfying the condition: there exists a constant C such that the inequality $\int_M e^{-\alpha\varphi} \leq C$ holds for all the ω -admissible functions φ with $\sup \varphi \geq 0$ (resp. with $\varphi(P_i) \geq 0$ for $1 \leq i \leq n$). Suppose in addition that the orbit of each P_i under the action of G has positive measure. Then $\alpha(\omega) = \alpha_{\Delta_n}(\omega)$.*

We first establish a few lemmas which will be useful for the proof.

Lemma 3.2. *Let $(\varphi_n)_{n \geq 0}$ be a sequence of admissible functions with nonnegative maxima. Then there exists a subset Ω of M , with $\mu_M(\Omega) = \mu_M(M)$, and a subsequence φ_{n_k} of φ_n , such that for every p in Ω , the sequence $(\varphi_{n_k}(p))_{k \geq 0}$ has a finite lower bound (depending on p).*

Proof. It is sufficient to assume that φ_n has null maxima. Let Q_n be a point such that $\varphi_n(Q_n)$ vanishes. Green's formula runs as follows:

$$\varphi_n(Q_n) = \frac{1}{V} \int_M \varphi_n + \int_M G(Q_n, R) \Delta \varphi_n(R) dV(R),$$

with $G(Q, R) \geq 0$ and $\int_M G(Q, R) dV(R) = C$, where C is a positive constant (see [Au4]). Since φ_n is admissible, $\Delta \varphi_n$ is less than m , m being the dimension of M . Thus $\int_M |\varphi_n| \leq C m V$. Furthermore, $\int_M \Delta \varphi_n = 0$, so $\int_M |\Delta \varphi_n| = 2 \int_{\{\Delta \varphi_n > 0\}} \Delta \varphi_n \leq 2mV$.

For every Q in M , we have $\nabla \varphi_n(Q) = \int_M \nabla_Q G(Q, R) \Delta \varphi_n(R) dv(R)$, so that

$$\int_M |\nabla \varphi_n| \leq \int_M \left[\int_M |\nabla_Q G(Q, R)| dv(Q) \right] |\Delta \varphi_n(R)| dv(R) \leq 2m\tilde{C}V,$$

since $\int_M |\nabla_Q G(Q, R)| dv(Q)$ is a continuous, hence a bounded function on M . Thus $(\varphi_n)_{n \geq 0}$ is bounded in the Sobolev space $H^{1,1}(M)$. By Kondrakov's theorem, we can extract from $(\varphi_n)_{n \geq 0}$ a subsequence which converges in $L^1(M)$, and after an other extraction we can suppose that this sequence converges almost everywhere to a function φ of $L^1(M)$. Since φ is finite almost everywhere, we get the result. \square

Lemma 3.3. *Let $(\varphi_n)_{n \geq 0}$ be a sequence of admissible functions with nonnegative maxima and suppose that there exists a compact group G of holomorphic isometries of M such that the orbit of each P_i has positive measure. Let $\Phi : G \rightarrow \mathbb{R} \cup \{-\infty\}$ be the map defined by $\Phi(g) = \inf_{\Delta_n} \inf_{k \geq 0} (\varphi_k \circ g)$. Then there exists g in G such that $\Phi(g)$ is finite.*

Proof. Suppose that $\Phi \equiv -\infty$. For $i = 1, \dots, n$, let A_i be the set of the g in G such that $\inf_{k \geq 0} (\varphi_k \circ g)(P_i) = -\infty$. The sets A_i are measurable and $\cup_{i=1}^n A_i = G$, so there exists i such that A_i has positive measure. From Lemma (3.2), $A_i.P_i$ is a subset of Ω^c . Since Ω and M have the same measure, the measure of $A_i.P_i$ vanishes. Let u_i be the map from G to M which sends g to $g(P_i)$. Then u_i has constant rank on G . Indeed, $u_i \circ L(g) = \sigma_g \circ u_i$, where $L(g)$ is the left translation by g and σ_g the map from M to M which sends x to $g.x$. Since $G.P_i$ has positive measure, u_i is a submersion on G , so that $u_i(A_i)$ has positive measure. This is a contradiction since $u_i(A_i) = A_i.P_i$. \square

We can now prove Theorem (3.1).

Proof. It is clear that $\alpha(\omega) \leq \alpha_{\Delta_n}(\omega)$. Conversely, let $\varepsilon > 0$. There exists a sequence $(\varphi_n)_{n \geq 0}$ of admissible functions with positive maxima such that $\int_M e^{-(\alpha(\omega) + \varepsilon)\varphi_k}$ goes to infinity as k goes to infinity. Replacing φ_n by $\varphi_n - \sup \varphi_n$, we can take $\sup \varphi_n = 0$. First we apply Lemma(3.2). For the sake of simplicity, we take $\varphi_{n_k} = \varphi_k$. From Lemma (3.3), there exists an element g in G such that $\Phi(g)$ is finite; we define Ψ_k by $\Psi_k = \varphi_k \circ g - \Phi(g)$. Since g is an isometry, Ψ_k is ω -admissible, and from the very definition of Φ , $\Psi_k(P_i)$ is nonnegative. Furthermore, $\int_M e^{-(\alpha(\omega) + \varepsilon)\Psi_k} = e^{(\alpha(\omega) + \varepsilon)\Phi(g)} \int_M e^{-(\alpha(\omega) + \varepsilon)\varphi_k}$. This proves that $\int_M e^{-(\alpha(\omega) + \varepsilon)\Psi_k}$ goes to infinity as k goes to infinity. Then, $\alpha_{\Delta_n}(\omega) \leq \alpha(\omega) + \varepsilon$. This inequality holds for every positive ε , and so $\alpha_{\Delta_n}(\omega) \leq \alpha(\omega)$. \square

3.2. Tian's invariant on a product. For a Kähler form ω on a compact Kähler manifold M , $\alpha(\omega)$ is defined as in Theorem (3.1).

Proposition 3.4. *Let $(M_i)_{1 \leq i \leq n}$ be compact Kähler manifolds with metric forms $(\omega_i)_{1 \leq i \leq n}$. We endow the product $M_1 \times \cdots \times M_n$ with the metric $\omega_1 \oplus \cdots \oplus \omega_n$. Then $\alpha(\omega_1 \oplus \cdots \oplus \omega_n) = \inf_{1 \leq i \leq n} \alpha(\omega_i)$.*

Proof. It suffices to make the proof when $n = 2$, the general result will follow by induction.

(1) Suppose that $\alpha(\omega_1) \leq \alpha(\omega_2)$, and let $\varepsilon > 0$. There exists a sequence $(\varphi_n)_{n \geq 0}$ of ω_1 -admissible functions on M_1 with positive maxima such that $\int_{M_1} e^{-(\alpha(\omega_1) + \varepsilon)\varphi_n}$ goes to infinity when n goes to infinity. We define ψ_n on $M_1 \times M_2$ by $\psi_n(m_1, m_2) = \varphi_n(m_1)$. Thus ψ_n is $(\omega_1 \oplus \omega_2)$ -admissible on $M_1 \times M_2$, with positive maximum, and $\int_{M_1 \times M_2} e^{-(\alpha(\omega_1) + \varepsilon)\psi_n} = V(M_2) \int_{M_1} e^{-(\alpha(\omega_1) + \varepsilon)\varphi_n}$, so that $\int_{M_1 \times M_2} e^{-(\alpha(\omega_1) + \varepsilon)\psi_n}$ goes to infinity when n goes to infinity. We have therefore $\alpha(\omega_1 \oplus \omega_2) \leq \alpha(\omega_1) + \varepsilon$. This yields $\alpha(\omega_1 \oplus \omega_2) \leq \alpha(\omega_1)$.

(2) Let us now prove the opposite inequality. Let α be a real number such that $\alpha < \inf(\alpha(\omega_1), \alpha(\omega_2))$ and φ an $(\omega_1 \oplus \omega_2)$ -admissible function on $M_1 \times M_2$. If m_2 is in M_2 , the function which sends m_1 to $\varphi(m_1, m_2)$ is ω_1 -admissible. The same holds for M_1 . Let (u, v) in $M_1 \times M_2$ be such that $\varphi(u, v) \geq 0$. Then

$$\begin{aligned} \int_{M_1 \times M_2} e^{-\alpha\varphi(m_1, m_2)} dV_1 dV_2 &= \int_{M_1} e^{-\alpha\varphi(m_1, v)} \left(\int_{M_2} e^{-\alpha[\varphi(m_1, m_2) - \varphi(m_1, v)]} dV_2 \right) dV_1 \\ &\leq C_2 \int_{M_1} e^{-\alpha\varphi(m_1, v)} dV_1 \leq C_1 C_2. \end{aligned}$$

Thus, $\alpha \leq \alpha(\omega_1 \oplus \omega_2)$ and we get $\inf(\alpha(\omega_1), \alpha(\omega_2)) \leq \alpha(\omega_1 \oplus \omega_2)$. \square

3.3. Tian's invariant on $G_{p,q}(\mathbb{C})$. Since there is a natural duality isomorphism between $G_{p,q}(\mathbb{C})$ and $G_{q,p}(\mathbb{C})$, we can assume that $p \leq q$ without loss of generality.

3.3.1. Imbedding of $\{\mathbb{P}^1(\mathbb{C})\}^p$ into $G_{p,q}(\mathbb{C})$ when $p \leq q$. For w in $\mathbb{C}^{p(q-1)}$, $w = (w_{i,j})_{\substack{1 \leq i \leq q \\ 1 \leq j \leq p \\ i \neq j}}$,

we define the map $\tilde{\rho}_w$ from $\{\mathbb{C}^2 \setminus (0,0)\}^p$ to $M_{p+q,p}(\mathbb{C})$ by

$$\tilde{\rho}_w((\lambda_i, \mu_i)_{1 \leq i \leq p}) = \begin{cases} \lambda_i \delta_{ij} & \text{if } i \leq p \\ w_{i-p,j} \lambda_j & \text{if } i > p \text{ and } i \neq j+p \\ \mu_i & \text{if } i > p \text{ and } i = j+p \end{cases}$$

We make, for $p+1 \leq i \leq p+q$, the following row transformations: $L_i \leftarrow L_i - \sum_{\substack{1 \leq j \leq p \\ i \neq j+p}} w_{i-p,j} L_j$.

We get a matrix $(c_{ij})_{\substack{1 \leq i \leq p+q \\ 1 \leq j \leq p}}$ with $c_{ij} = \delta_{ij} \lambda_i$ if $1 \leq i \leq p$ and $c_{ij} = \delta_{i-p,j} \mu_j$ if $p+1 \leq i \leq p+q$, which has rank p . $\tilde{\rho}_w$ induces a map from $\{\mathbb{P}^1(\mathbb{C})\}^p$ into $G_{p,q}(\mathbb{C})$ as shown on the following diagram, where γ is the projection of the principal fiber bundle $\{\mathbb{C}^2 \setminus (0,0)\}^p$ onto $\{\mathbb{P}^1(\mathbb{C})\}^p$. Remark that $\tilde{\rho}_w$ sends $[0,1] \times \cdots \times [0,1]$ onto $\pi(A)$, where $m_{\{p+1, \dots, 2p\}}(A) = I^{(p)}$ and $m_{\{p+1, \dots, 2p\}^c}(A) = 0^{(q \times p)}$.

$$\begin{array}{ccc} \{\mathbb{C}^2 \setminus (0,0)\}^p & \xrightarrow{\tilde{\rho}_w} & M^*(p+q, p) \\ \gamma \downarrow & & \downarrow \pi \\ \{\mathbb{P}^1(\mathbb{C})\}^p & \xrightarrow{\rho_w} & G_{p,q}(\mathbb{C}) \end{array}$$

We have

$$\begin{aligned} (\pi \circ \tilde{\rho}_w)^*(\mathcal{G}_{p,q}) &= i \partial \bar{\partial} \log \left(\det({}^t \tilde{\rho}_w \bar{\rho}_w) \right) \\ &= i \partial \bar{\partial} \log \left(\frac{\det({}^t \tilde{\rho}_w \bar{\rho}_w)}{\prod_{k=1}^p (|\lambda_k|^2 + |\mu_k|^2)} \right) + \sum_{k=1}^p i \partial \bar{\partial} \log (|\lambda_k|^2 + |\mu_k|^2) \\ &= i \partial \bar{\partial} \log \tilde{\Phi} + \gamma^*(FS_1 \oplus \cdots \oplus FS_1), \end{aligned}$$

where FS_1 is the Fubini-Study metric on $\mathbb{P}^1(\mathbb{C})$. $\tilde{\Phi}$ is invariant by the action of the structural group $\mathbb{C}^* \times \cdots \times \mathbb{C}^*$, so it induces a map Φ from $\{\mathbb{P}^1(\mathbb{C})\}^p$ into \mathbb{C} . Note that $\Phi([0,1] \times \cdots \times [0,1]) = 1$. Then $(\pi \circ \tilde{\rho}_w)^*(\mathcal{G}_{p,q}) = \pi^*(i \partial \bar{\partial} \log \Phi + FS_1 \oplus \cdots \oplus FS_1)$, so that $\rho_w^*(\mathcal{G}_{p,q}) = i \partial \bar{\partial} \log \Phi + FS_1 \oplus \cdots \oplus FS_1$.

3.3.2. *Lower bound of $\alpha(\mathcal{G}_{p,q})$.* For I in \mathcal{I} , we define P_I by $m_I(P_I) = I^{(p)}$ and $m_{I^c}(P_I) = 0^{(q \times p)}$. If $n = \binom{p+q}{p}$, we set $\Delta_n = \{P_I\}_{I \in \mathcal{I}}$. Since $U(p+q)$ is a transitive group of holomorphic isometries of $G_{p,q}(\mathbb{C})$, we know from proposition (3.1), that $\alpha(\mathcal{G}_{p,q}) = \alpha_{\Delta_n}(\mathcal{G}_{p,q})$. We set $I = \{p+1, \dots, 2p\}$. Let φ be an admissible function on $G_{p,q}(\mathbb{C})$, nonnegative on Δ_n . The last equality of the precedent section shows that the function $\varphi \circ \rho_w + \log \Phi$ is $(FS_1 \oplus \dots \oplus FS_1)$ -admissible for every w in $\mathbb{C}^{p(q-1)}$. Furthermore, $(\varphi \circ \rho_w + \log \Phi)$ sends $[0, 1] \times \dots \times [0, 1]$ to the nonnegative number $\varphi(P_I)$. It is known that $\alpha(FS_1) = 1$ (see [Au3]). Proposition (3.4) yields $\alpha(FS_1 \oplus \dots \oplus FS_1) = 1$.

Let α be a real number such that $\alpha < 1$. There exists a constant C , independent of φ , such that $\int_{\{\mathbb{P}^1(\mathbb{C})\}^p} e^{-\alpha \varphi \circ \rho_w} \Phi^{-\alpha} \leq C$. We define the map F_I from $\pi(U_I)$ to \mathbb{R}_+ by $F_I(Z_I) = \det(Id + {}^t Z_I \bar{Z}_I)$. On $\{\mathbb{P}^1(\mathbb{C})\}^p$, we work with the coordinates μ_1, \dots, μ_p in the chart $\lambda_1 = \dots = \lambda_p = 1$. Thus

$$\Phi(\mu) = \frac{F_I \circ \rho_w(\mu)}{\prod_{k=1}^p (1 + |\mu_k|^2)}, \quad \text{so that} \quad \int_{\mu \in \mathbb{C}^p} e^{-\alpha \varphi \circ \rho_w(\mu)} \frac{dV_\mu(\mathbb{C}^p)}{\prod_{k=1}^p (1 + |\mu_k|^2)^{2-\alpha} (F_I \circ \rho_w(\mu))^\alpha} \leq C.$$

We have the inequality $\sum_{i=1}^q \sum_{j=1}^p |Z_{ij}|^2 \leq F_I(P_I)$. In particular, for every k in $\{1, \dots, p\}$, $1 + |\mu_k|^2 \leq F_I \circ \rho_w(\mu)$, and $F_I \circ \rho_w(\mu) \geq 1 + \sum_{\substack{1 \leq i \leq q \\ 1 \leq j \leq p \\ i \neq j}} |w_{ij}|^2$. Thus, for $\kappa > 0$ and $w \in \mathbb{C}^{p(q-1)}$,

$$\frac{\prod_{k=1}^p (1 + |\mu_k|^2)^{2-\alpha}}{(F_I \circ \rho_w(\mu))^{\kappa+p+q-\alpha}} \leq \frac{1}{(F_I \circ \rho_w(\mu))^{\kappa-p+q+\alpha(p-1)}} \leq \frac{1}{\left(1 + \sum_{\substack{1 \leq i \leq q \\ 1 \leq j \leq p \\ i \neq j}} |w_{ij}|^2\right)^\kappa} = \frac{1}{\left(1 + \|w\|^2\right)^\kappa}.$$

We have, according to Proposition (2.3),

$$\begin{aligned}
\int_{\pi(U_I)} \frac{e^{-\alpha\varphi}}{F_I^\kappa} &= \int_{w \in \mathbb{C}^{p(q-1)}} \int_{\mu \in \mathbb{C}^p} \frac{e^{-\alpha\varphi \circ \rho_w(\mu)}}{(F_I \circ \rho_w(\mu))^{\kappa+p+q}} dV_\mu(\mathbb{C}^p) dV_w(\mathbb{C}^{p(q-1)}) \\
&= \int_{w \in \mathbb{C}^{p(q-1)}} \int_{\mu \in \mathbb{C}^p} \left(\frac{e^{-\alpha\varphi \circ \rho_w(\mu)}}{\prod_{k=1}^p (1 + |\mu_k|^2)^{2-\alpha} (F_I \circ \rho_w(\mu))^\alpha} \right) \\
&\quad \times \frac{\prod_{k=1}^p (1 + |\mu_k|^2)^{2-\alpha}}{(F_I \circ \rho_w(\mu))^{\kappa+p+q-\alpha}} dV_\mu(\mathbb{C}^p) dV_w(\mathbb{C}^{p(q-1)}) \\
&= \int_{w \in \mathbb{C}^{p(q-1)}} \left(\int_{\mu \in \mathbb{C}^p} \frac{e^{-\alpha\varphi \circ \rho_w(\mu)}}{\prod_{k=1}^p (1 + |\mu_k|^2)^{2-\alpha} (F_I \circ \rho_w(\mu))^\alpha} dV_\mu(\mathbb{C}^p) \right) \times \frac{dV_w(\mathbb{C}^{p(q-1)})}{(1 + \|w\|^2)^\kappa} \\
&\leq C \int_{w \in \mathbb{C}^{p(q-1)}} \frac{dV_w(\mathbb{C}^{p(q-1)})}{(1 + \|w\|^2)^\kappa} \leq C' \quad \text{if } \kappa > p(q-1).
\end{aligned}$$

Thus, we obtain that for all I in \mathcal{I} , $\int_{\pi(U_I)} \frac{e^{-\alpha\varphi}}{F_I^\kappa} \leq C$, where C is independent of φ .

Since $G_{p,q}(\mathbb{C})$ is compact, there exists a family $(V_I)_{I \in \mathcal{I}}$ of open sets of $G_{p,q}(\mathbb{C})$ such that V_I is relatively compact in $\pi(U_I)$ for every $I \in \mathcal{I}$, and $\bigcup_{I \in \mathcal{I}} V_I = G_{p,q}(\mathbb{C})$. There exists $M > 0$ such that $F_I \leq M$ on V_I for every $I \in \mathcal{I}$. Thus

$$\int_{G_{p,q}(\mathbb{C})} e^{-\alpha\varphi} \leq \sum_{I \in \mathcal{I}} \int_{V_I} e^{-\alpha\varphi} \leq \sum_{I \in \mathcal{I}} M^\kappa \int_{V_I} \frac{e^{-\alpha\varphi}}{F_I^\kappa} \leq M^\kappa \sum_{I \in \mathcal{I}} \int_{\pi(U_I)} \frac{e^{-\alpha\varphi}}{F_I^\kappa} \leq C M^\kappa \binom{p+q}{p}.$$

We deduce that $\alpha(\mathcal{G}_{p,q}) \geq 1$.

3.3.3. Upper bound of $\alpha(\mathcal{G}_{p,q})$. We use here a method which can be found in [Re] for the complex projective space. Let I in \mathcal{I} . We define \tilde{K} from $M^*(p+q, p)$ to $\mathbb{P}^1(\mathbb{R})$ by the relation $\tilde{K}(M) = [|\det m_I(H)|^2, \det {}^t M \overline{M}]$. \tilde{K} is invariant by the action of the structural group $G_p(\mathbb{C})$, so it induces a C^∞ map K from $G_{p,q}(\mathbb{C})$ to $\mathbb{P}^1(\mathbb{R})$. Remark that $\psi = \log K$ is a Kähler potential on U_I for the metric $\mathcal{G}_{p,q}$.

Lemma 3.5. *There exists a decreasing sequence $(\varphi_n)_{n \geq 0}$ of admissible functions with positive maxima which converges pointwise to $-\psi$ on $\pi(U_I)$.*

Proof. We construct a decreasing sequence $(f_n)_{n \geq 0}$ of C^∞ convex functions on \mathbb{R}_+ satisfying the conditions $1 + f'_n > 0$, $f_n(x) = -(1 - 1/n)x$ for x in $[0, n]$ and $f_n(x) = -n$ for

$x \geq 2n$. Let y be an element of $\pi(U_I)^c$ and Ω_n the set of the elements x in $\pi(U_I)$ such that $\psi(x) > 2n$. Since $F_I(y) = [0, 1]$, there exists a neighborhood V of y such that the inequality $z > e^{2n}$ holds for every point $[1, z]$ in $F_I(V)$. Thus $V \cap \pi(U_I)$ is included in Ω_n . We have proved that $W_n = \Omega_n \cup \pi(U_I)^c$, so that W_n is an open neighborhood of $\pi(U_I)^c$. We define φ_n by $\varphi_n = f_n \circ \psi$ on $\pi(U_I)$ and $\varphi_n = -n$ on W_n . Thus φ_n is well defined and $\varphi_n(0) = 0$. It remains to show that φ_n is admissible on $\pi(U_I)$. We have

$$\left(\mathcal{G}_{p,q} + i \partial \bar{\partial} \varphi_n \right)_{\lambda \bar{\mu}} = \partial_{\lambda \bar{\mu}} \psi + \partial_{\lambda} (f'_n \circ \psi) \partial_{\bar{\mu}} \psi = (1 + f'_n \circ \psi) \partial_{\lambda \bar{\mu}} \psi + f''_n \circ \psi \partial_{\lambda} \psi \partial_{\bar{\mu}} \psi.$$

Hence the matrix of the metric $\mathcal{G}_{p,q} + i \partial \bar{\partial} \varphi_n$ is of the form $A + T$ where A is positive definite and T has rank one and positive trace. So $A + T$ is positive definite and we get the result. \square

Lemma 3.6. *Let n in \mathbb{N}^* and r a positive real number. Then*

$$\int_{\|X\| \leq r} \frac{dV_X(M_n(\mathbb{C}))}{|\det X|^2} = +\infty.$$

Proof. We can write

$$\int_{\|X\| \leq r} \frac{dV_X(M_n(\mathbb{C}))}{|\det X|^2} = \sum_{k=0}^{\infty} \int_{r/2^{k+1} \leq \|X\| \leq r/2^k} \frac{dV_X(M_n(\mathbb{C}))}{|\det X|^2}.$$

We put $Y = 2^k X$, so

$$\int_{r/2^{k+1} \leq \|X\| \leq r/2^k} \frac{dV_X(M_n(\mathbb{C}))}{|\det X|^2} = \int_{1/2 \leq \|Y\| \leq 1} \frac{dV_Y(M_n(\mathbb{C}))}{|\det Y|^2}.$$

The terms in the series are strictly positive and independent of k . The sum is therefore infinite. \square

We can now prove that $\alpha(\mathcal{G}_{p,q})$ is upper bounded by 1. Suppose that $\alpha(\mathcal{G}_{p,q}) > 1$.

Then there exists a positive C such that for every integer n , $\int_{\pi(U_I)} e^{-\varphi_n} \leq C$. Us-

ing Lemma (3.5) and monotonous convergence, $\int_{\pi(U_I)} F_I \leq C$. Since $\pi(U_I)^c$ has zero

measure, $\int_{G_{p,q}(\mathbb{C})} F_I \leq C$. Let \tilde{I} in \mathcal{I} be such that $I \cap \tilde{I} = \emptyset$ (this is possible since

$p \leq q$). We have $P_{\tilde{I}} \{m_I(P_{\tilde{I}})\}^{-1} = P_{\tilde{I}}$. Remark that $m_I(P_{\tilde{I}}) = m_I(Z_{\tilde{I}})$. Thus $\det(Id + {}^t Z_I \bar{Z}_I) = \det({}^t P_{\tilde{I}} \bar{P}_{\tilde{I}}) |\det m_I(Z_{\tilde{I}})|^{-2}$. For $\|Z_{\tilde{I}}\| \leq r$, $\det({}^t P_{\tilde{I}} \bar{P}_{\tilde{I}}) \leq M$, so that

$\int_{\|Z_{\tilde{I}}\| \leq r} \frac{dV_{Z_{\tilde{I}}}(M_{q,p}(\mathbb{C}))}{|\det m_I(Z_{\tilde{I}})|^2} < +\infty$. Integrating over the remaining variables $(Z_{ij})_{i \in \tilde{I}^c \cap I^c}$ yields

$\int_{\|Z\| \leq r} \frac{dV_Z(M_p(\mathbb{C}))}{|\det Z|^2} < +\infty$, which is in contradiction with the result of Lemma (3.6).
Thus we obtain $\alpha(\mathcal{G}_{p,q}) \leq 1$.

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